MINIMAL ENERGY SOLUTIONS AND INFINITELY MANY BIFURCATING BRANCHES FOR A CLASS OF SATURATED NONLINEAR SCHRÖDINGER SYSTEMS

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Abstract. We prove a conjecture which was recently formulated by Maia, Montefusco, Pellacci saying that minimal energy solutions of the saturated nonlinear Schrödinger system

$$-\Delta u + \lambda_1 u = \frac{\alpha u(\alpha u^2 + \beta v^2)}{1 + s(\alpha u^2 + \beta v^2)} \qquad \text{in } \mathbb{R}^n,$$
$$-\Delta v + \lambda_2 v = \frac{\beta v(\alpha u^2 + \beta v^2)}{1 + s(\alpha u^2 + \beta v^2)} \qquad \text{in } \mathbb{R}^n,$$

are necessarily semitrivial whenever $\alpha, \beta, \lambda_1, \lambda_2 > 0$ and $0 < s < \max\{\frac{\alpha}{\lambda_1}, \frac{\beta}{\lambda_2}\}$ except for the symmetric case $\lambda_1 = \lambda_2, \alpha = \beta$. Moreover it is shown that for most parameter samples $\alpha, \beta, \lambda_1, \lambda_2$ there are infinitely many branches containing seminodal solutions which bifurcate from a semitrivial solution curve parametrized by s.

1. Introduction

In this paper we intend to continue the study on nonlinear Schrödinger systems for saturated optical materials which was recently initiated by Maia, Montefusco and Pellacci [11]. In their paper the following system of elliptic partial differential equations

$$(1)\qquad \begin{aligned} -\Delta u + \lambda_1 u &= \frac{\alpha u(\alpha u^2 + \beta v^2)}{1 + s(\alpha u^2 + \beta v^2)} \qquad \text{in } \mathbb{R}^n,\\ -\Delta v + \lambda_2 v &= \frac{\beta v(\alpha u^2 + \beta v^2)}{1 + s(\alpha u^2 + \beta v^2)} \qquad \text{in } \mathbb{R}^n,\end{aligned}$$

was suggested in order to model the interaction of two pulses within the optical material under investigation. Here, the parameters satisfy $\lambda_1, \lambda_2, \alpha, \beta, s > 0$ and $n \in \mathbb{N}$. One way to find classical fully nontrivial solutions of (1) is to use variational methods. The Euler functional $I_s : H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \to \mathbb{R}$ associated to the system (1) is given by

$$(2)\qquad \begin{aligned} I_s(u,v) &:= \frac{1}{2}\Big(\|u\|_{\lambda_1}^2 + \|v\|_{\lambda_2}^2 - \frac{\alpha}{s}\|u\|_2^2 - \frac{\beta}{s}\|v\|_2^2\Big) + \frac{1}{2s^2}\int_{\mathbb{R}^n} \ln(1 + s(\alpha u^2 + \beta z^2))\\ &= \frac{1}{2}(\|u\|_{\lambda_1}^2 + \|v\|_{\lambda_2}^2) - \frac{1}{2s^2}\int_{\mathbb{R}^n} g(sZ)\end{aligned}$$

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where $Z(x) := \alpha u(x)^2 + \beta v(x)^2$ and $g(z) := z - \ln(1+z)$ for all $z \geq 0$. The symbol $\|\cdot\|_2$ denotes the standard norm on $L^2(\mathbb{R}^n)$ and the norms $\|\cdot\|_{\lambda_1}, \|\cdot\|_{\lambda_2}$ are defined via

$$\|u\|_{\lambda_1} := \Big(\int_{\mathbb{R}^n} |\nabla u|^2 + \lambda_1 u^2\Big)^{1/2}, \qquad \|v\|_{\lambda_2} := \Big(\int_{\mathbb{R}^n} |\nabla v|^2 + \lambda_2 v^2\Big)^{1/2}.$$

Since we are interested in minimal energy solutions (i.e. ground states) for (1) the ground states u_s, v_s of the scalar problems associated to (1) turn out to be of particular importance. These are positive radially symmetric and radially decreasing smooth functions satisfying

$$-\Delta u_s + \lambda_1 u_s = \frac{\alpha^2 u_s^3}{1+s\alpha u_s^2} \quad \text{in } \mathbb{R}^n, \qquad -\Delta v_s + \lambda_2 v_s = \frac{\beta^2 v_s^3}{1+s\beta v_s^2} \quad \text{in } \mathbb{R}^n. \tag{3}$$

Since we will encounter these solutions many times let us recall some facts from the literature. Existence of positive finite energy solutions u_s, v_s of (3) for parameters $0 < s < \frac{\alpha}{\lambda_1}$ respectively $0 < s < \frac{\beta}{\lambda_2}$ can be deduced from Theorem 2.2 in [20] in case $n \geq 3$ or Theorem 1(i) in [5] for $n \geq 2$. In case $n = 1$ the positive functions u_s, v_s are given by $u_s(x) = u_s(-x), v_s(x) = v_s(-x)$ for all $x \in \mathbb{R}$ and

$$u_s|_{[0,\infty)}^{-1}(z) = \int_z^{u_s(0)} \Big(\frac{1}{\lambda_1 x^2 - s^{-2} g(s\alpha x^2)}\Big)^{1/2} dx, \qquad \text{for } z \in (0, u_s(0)],$$

$$v_s|_{[0,\infty)}^{-1}(z) = \int_z^{v_s(0)} \Big(\frac{1}{\lambda_2 x^2 - s^{-2} g(s\beta x^2)}\Big)^{1/2} dx \qquad \text{for } z \in (0, v_s(0)]$$

where $u_s(0), v_s(0) > 0$ are uniquely determined by

$$\lambda_1 u_s(0)^2 - s^{-2} g(s\alpha u_s(0)^2) = \lambda_2 v_s(0)^2 - s^{-2} g(s\beta v_s(0)^2) = 0. \tag{4}$$

As in the explicit one-dimensional case it is known also in the higher-dimensional case that u_s, v_s are radially symmetric, see Theorem 2 in [6]. Finally, the uniqueness of u_s, v_s follows from Theorem 1 in [18] in case $n \geq 3$ and from Theorem 1 in [13] in case $n = 2$. The uniqueness result for $n = 1$ is a direct consequence of the existence proof we gave above.

In this paper we strengthen the results obtained by Maia, Montefusco, Pellacci [11] concerning ground state solutions and (component-wise) positive solutions of (1), so let us shortly comment on their achievements. In Theorem 3.7 of their paper they proved the existence of nonnegative radially symmetric and nonincreasing ground state solutions of (1) for all $n \geq 2$ and parameter values $0 < s < \max\{\frac{\alpha}{\lambda_1}, \frac{\beta}{\lambda_2}\}$ where the upper bound for s is in fact optimal by Lemma 3.2 in the same paper. It was conjectured that each of these ground states is semitrivial except for the special case $\alpha = \beta, \lambda_1 = \lambda_2$ where the totality of ground state solutions is known in a somehow explicit way, see Theorem 2.1 in [11] or Theorem 1 (i) below. In [11] this conjecture was proved for parameters $s \geq \min\{\frac{\alpha}{\lambda_1}, \frac{\beta}{\lambda_2}\}$, see Theorem 3.15 and Theorem 3.17. Our first result shows that the full conjecture is true even in the case $n = 1$ which was left aside in [11].

Theorem 1. *Let $n \in \mathbb{N}, \alpha, \beta, \lambda_1, \lambda_2 > 0$ and $0 < s < \max\{\frac{\alpha}{\lambda_1}, \frac{\beta}{\lambda_2}\}$. Then the following holds:*

(i) In case $\alpha = \beta$ and $\lambda_1 = \lambda_2$ all ground states of (1) *are given by* $(\cos(\theta)u_s, \sin(\theta)v_s)$ *for $\theta \in [0, 2\pi)$.*

(ii) In case $\alpha \neq \beta$ or $\lambda_1 \neq \lambda_2$ every ground state solution of (1) *is semitrivial.*

The proof of this result will we presented in section 2. Our approach is based on a suitable min-max characterization of the Mountain pass level associated to (1) involving a fibering map technique as in [12]. This method even allows to give an alternative proof for the existence of a ground state solution of (1) which is significantly shorter than the one presented in [11] and which moreover incorporates the case $n = 1$, see Proposition 1. More importantly this approach yields the optimal result.

In view of Theorem 1 it is natural to ask how the existence of fully nontrivial solutions of (1) can be proved. In [11] Maia, Montefusco, Pellacci found necessary conditions and sufficient conditions for the existence of positive solutions of (1) which, however, partly contradict each other. For instance, Theorem 3.21 in [11] claims that positive solutions exist for parameters $\alpha = \beta, \lambda_1 \neq \lambda_2$ and $s > 0$ sufficiently small contradicting the nonexistence result from Theorem 3.10. The error leading to this contradiction is located on page 338, line 13 in [11] where the number $\frac{\lambda_2}{s}$ must be replaced by $\lambda_2 s$ which destroys the results from Theorem 3.19 and Theorem 3.21. Our approach to finding positive solutions and, more generally, seminodal solutions of (1) is to apply bifurcation theory to the semitrivial solution branches

$$\mathcal{T}_1 := \Big\{(0, v_s, s) : 0 < s < \frac{\beta}{\lambda_2}\Big\}, \qquad \mathcal{T}_2 := \Big\{(u_s, 0, s) : 0 < s < \frac{\alpha}{\lambda_1}\Big\}$$

which was motivated by the papers of Ostrovskaya, Kivshar [14] and Champneys, Yang [3]. In the case $n = 1$ and $\lambda_1 = 1, \lambda_2 = \omega^2 \in (0, 1), \alpha = \beta = 1$ they numerically detected a large number of solution branches emanating from $\mathcal{T}_2$ which consist of seminodal solutions. Moreover, they conjectured that the bifurcation points on $\mathcal{T}_2$ accumulate near $s = 1$, see page 2184 ff. in [3]. Our results confirm these observations. For simplicity we will only discuss the bifurcations from $\mathcal{T}_2$ since the corresponding analysis for $\mathcal{T}_1$ is the same up to interchanging the roles of λ_1, λ_2 and α, β. Investigating the linearized problems associated to (1) near $(u_s, 0, s)$ for parameters close to the boundary of the parameter interval $(0, \frac{\alpha}{\lambda_1})$ we prove the existence of infinitely many bifurcating branches containing fully nontrivial solutions of a certain nodal pattern. Despite the fact that the question whether fully nontrivial solutions bifurcate from $\mathcal{T}_1, \mathcal{T}_2$ makes perfect sense for all space dimensions $n \in \mathbb{N}$ our bifurcation result is restricted to $n \in \{1, 2, 3\}$. Later we will comment on this issue in more detail, see Remark 2. In order to formulate our bifurcation result let us define the positive numbers $\bar{\mu}_k$ to be the k-th eigenvalues of the linear compact self-adjoint operators $\phi \mapsto (-\Delta + \lambda_2)^{-1}(\alpha\beta u_0^2 \phi)$ mapping $H_r^1(\mathbb{R}^n)$ to itself where u_0 denotes the positive ground state solution of the first equation in (3) for $s = 0$. By Sturm-Liouville theory we know that these eigenvalues are simple and that they satisfy

$$\bar{\mu}_0 > \bar{\mu}_1 > \bar{\mu}_2 > \ldots > \bar{\mu}_k \to 0^+ \quad \text{as } k \to \infty.$$

Deferring some more or less standard notational convention to a later moment we come to the statement of our result.

Theorem 2. *Let $n \in \{1, 2, 3\}$ and let $\alpha, \beta, \lambda_1, \lambda_2 > 0$ and $k_0 \in \mathbb{N}_0$ satisfy*

$$\frac{\lambda_2}{\lambda_1} < \frac{\beta}{\alpha} \qquad \textit{and} \qquad \overline{\mu}_{k_0} < 1.$$

Then there is an increasing sequence $(s_k)_{k\geq k_0}$ of positive numbers converging to $\frac{\alpha}{\lambda_1}$ such that continua $\mathcal{C}_k \subset \mathcal{S}$ containing $(0,k)-$nodal solutions of (1) emanate from $\mathcal{T}_2$ at $s = s_k$ $(k \geq k_0)$. In case $k_0 = 0$ we have $\lambda_1 > \lambda_2$ and there is a $C > 0$ such that all positive solutions $(u,v,s) \in \mathcal{C}_0$ with $s \geq 0$ satisfy

$$\|u\|_{\lambda_1} + \|v\|_{\lambda_2} < C \qquad \text{and} \qquad s < \frac{\alpha-\beta}{\lambda_1-\lambda_2} < \frac{\alpha}{\lambda_1}. \tag{5}$$

In case $n \in \{2,3\}$ we can estimate $\bar{\mu}^0$ from above in order to obtain a sufficient condition for the conclusions of Theorem 2 to hold for $k_0 = 0$. This estimate leading to Corollary 1 is based on the Courant-Fischer min-max-principle and Hölder's inequality. In the one-dimensional case the values of all eigenvalues $\bar{\mu}_k$ are explicitly known which results in Corollary 2.

Corollary 1. *Let $n \in \{2,3\}$. Then the conclusions from Theorem 2 are true for $k_0 = 0$ if*

$$\frac{\lambda_2}{\lambda_1} < \frac{\beta}{\alpha} < \Big(\frac{\lambda_2}{\lambda_1}\Big)^{\frac{4-n}{4}}. \tag{6}$$

Corollary 2. *Let $n = 1$. Then the conclusions from Theorem 2 are true in case*

$$\frac{\lambda_2}{\lambda_1} < \frac{\beta}{\alpha} < \frac{1}{2}\Big(\sqrt{\frac{\lambda_2}{\lambda_1}} + 2k_0\Big)\Big(\sqrt{\frac{\lambda_2}{\lambda_1}} + 2k_0 + 1\Big). \tag{7}$$

Remark 1. *As we mentioned above one can find sufficient criteria for the existence of $(k,0)$-nodal solutions bifurcating from $\mathcal{T}_1$ by inversing the roles of λ_1, λ_2 and α, β in the statement of Theorem 2 as well as in its Corollaries.*

Theorem 2 gives rise to many questions which would be interesting to solve in the future. A list of open problems is provided in section 5. Before going on with the proof of our results let us clarify the notation which we used in Theorem 2. The set $\mathcal{S} \subset X \times \mathbb{R}$ is the closure of all solutions of (1) which do not belong to $\mathcal{T}_2$ and a subset of $\mathcal{S}$ is called a continuum if it is a maximal connected set within $\mathcal{S}$. Finally, a fully nontrivial solution (u,v) of (1) is called (k,l)-nodal if both component functions are radially symmetric and u has precisely $k+1$ nodal annuli and v has precisely $l+1$ nodal annuli. In other words, since double zeros can not occur, (u,v) is (k,l) nodal if the radial profiles of u respectively v have precisely k respectively l zeros.

2. Proof of Theorem 1

According to the assumptions of Theorem 1 we will assume throughout this section that the numbers $\lambda_1, \lambda_2, \alpha, \beta$ are positive, that s lies between 0 and $\max\{\frac{\alpha}{\lambda_1}, \frac{\beta}{\lambda_2}\} =: s^*$ and that the space dimension is an arbitrary natural number. Furthermore, we define the energy levels

$$c_s = \inf\Big\{I_s(u,v) : (u,v) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \text{ solves } (1), (u,v) \neq (0,0)\Big\},$$
$$c_s^* = \inf\Big\{I_s(u,v) : (u,v) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \text{ solves } (1), u = 0, v \neq 0 \text{ or } u \neq 0, v = 0\Big\}.$$

The first step towards the proof of Theorem 1 is a more suitable min-max-characterization of the least energy level c_s of (1) which, as in [12], gives rise to a simple proof for the existence of a ground state. To this end we introduce the Nehari manifold

$$c_{\mathcal{N}_s} := \inf_{\mathcal{N}_s} I_s, \qquad \mathcal{N}_s := \{(u,v) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) : (u,v) \neq (0,0) \text{ and } I_s'(u,v)[(u,v)] = 0\}.$$

Proposition 1. *The value*

$$c_s = c_{\mathcal{N}_s} = \inf_{(u,v)\neq(0,0)} \sup_{r>0} I_s(\sqrt{r}u, \sqrt{r}v). \tag{8}$$

is attained at a radially symmetric and radially noninreasing ground state of (1).

Proof. From the equations (3.15),(3.52) in [11] we get $c_s = c_{\mathcal{N}_s}$, so let us prove the second equation in (8). For every fixed $u, v \in H^1(\mathbb{R}^n)$ satisfying $(u,v) \neq (0,0)$ we set

$$\beta(r) := I_s(\sqrt{r}u, \sqrt{r}v) = \frac{r}{2}(\|u\|_{\lambda_1}^2 + \|v\|_{\lambda_2}^2) - \frac{1}{2s^2}\int_{\mathbb{R}^n} g(rsZ)$$

so that $(\sqrt{r}u, \sqrt{r}v) \in \mathcal{N}_s$ holds for $r > 0$ if and only if $\beta'(r) = 0$. Since β is smooth and strictly concave with $\beta'(0) > 0$ a critical point of β is uniquely determined and it is a maximizer (whenever it exists). Since the supremum of β is $+\infty$ when there is no maximizer of β we obtain

$$c_{\mathcal{N}_s} = \inf_{\mathcal{N}_s} I_s = \inf_{(u,v)\neq(0,0)} \sup_{r>0} I_s(\sqrt{r}u, \sqrt{r}v)$$

which proves the formula (8).

Due to $0 < s < \max\{\frac{\alpha}{\lambda_1}, \frac{\beta}{\lambda_2}\}$ we can find a semitrivial function $(u,v) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ satisfying $\|u\|_{\lambda_1}^2 + \|v\|_{\lambda_2}^2 < \frac{\alpha}{s}\|u\|_2^2 + \frac{\beta}{s}\|v\|_2^2$ which implies $c_s < \infty$ according to (8). So let (u_k, v_k) be a minimizing sequence in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ satisfying $\sup_{r>0} I_s(\sqrt{r}u_k, \sqrt{r}v_k) \to c_s$ as $k \to \infty$. Using the classical Polya-Szegö-inequality and the extended Hardy-Littlewood inequality

$$\int_{\mathbb{R}^n} \ln\Big(1 + rs(\alpha u_k^2 + \beta v_k^2)\Big) \geq \int_{\mathbb{R}^n} \ln\Big(1 + rs(\alpha u_k^{*2} + \beta v_k^{*2})\Big) \qquad \text{for all } r > 0$$

for the spherical rearrangement taken from Theorem 2.2 in [1] we may assume u_k, v_k to be radially symmetric and radially decreasing. Since the function $g(z) = z - \ln(1+z)$ strictly increases on $(0,\infty)$ from 0 to $+\infty$ we may moreover assume that (u_k, v_k) are rescaled in such a way that the equality $\frac{1}{2s^2}\int_{\mathbb{R}^n} g(sZ_k) = 1$ holds for $Z_k := \alpha u_k^2 + \beta v_k^2$. The inequality

$$c_s + o(1) = \lim_{k\to\infty} \sup_{r>0} I_s(\sqrt{r}u_k, \sqrt{r}v_k) \geq \limsup_{k\to\infty} I_s(u_k, v_k) = \frac{1}{2}\limsup_{k\to\infty}(\|u_k\|_{\lambda_1}^2 + \|v_k\|_{\lambda_2}^2) - 1$$

implies that the sequence (u_k, v_k) is bounded in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$. Using the uniform decay rate and the resulting compactness properties of radially decreasing functions bounded in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ (apply for instance Compactness Lemma 2 in [19]) we may take a subsequence again denoted by (u_k, v_k) such that $(u_k, v_k) \rightharpoonup (u,v)$ in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$, pointwise always everywhere and

$$\frac{1}{2s^2}\int_{\mathbb{R}^n} g(rsZ) = \lim_{k\to\infty} \frac{1}{2s^2}\int_{\mathbb{R}^n} g(rsZ_k) \qquad \text{for all } r > 0.$$

From this we infer $\frac{1}{2s^2}\int_{\mathbb{R}^n} g(sZ) = 1$ and thus $(u,v)\neq(0,0)$. Hence, we obtain

$$\begin{aligned} c_s &= \lim_{k\to\infty}\sup_{\rho>0} I_s(\sqrt{\rho}u_k,\sqrt{\rho}v_k) \\ &\geq \limsup_{k\to\infty}\Big(\frac{r}{2}(\|u_k\|_{\lambda_1}^2+\|v_k\|_{\lambda_2}^2)-\frac{1}{2s^2}\int_{\mathbb{R}^n} g(rsZ_k)\Big) \\ &\geq \frac{r}{2}(\|u\|_{\lambda_1}^2+\|v\|_{\lambda_2}^2)-\frac{1}{2s^2}\int_{\mathbb{R}^n} g(rsZ) \\ &= I_s(\sqrt{r}u,\sqrt{r}v) \qquad\qquad \text{for all } r>0 \end{aligned}$$

so that (u,v) is a nontrivial radially symmetric and radially decreasing minimizer. Taking for r the maximizer of the map $r\mapsto I_s(\sqrt{r}u,\sqrt{r}v)$ we obtain the ground state solution $(\bar{u},\bar{v}) := (\sqrt{r}u,\sqrt{r}v)$ having the properties we claimed to hold. Indeed, the Nehari manifold may be rewritten as $\mathcal{N}_s=\{(u,v)\in H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n) : (u,v)\neq(0,0), H(u,v)=0\}$ for

$$H(u,v) := I_s'(u,v)[(u,v)] = \|u\|_{\lambda_1}^2+\|v\|_{\lambda_2}^2-\int_{\mathbb{R}^n}\frac{Z^2}{1+sZ}$$

so that the Lagrange multiplier rule applies due to

$$H'(u,v)[(u,v)] = 2(\|u\|_{\lambda_1}^2+\|v\|_{\lambda_2}^2)-\int_{\mathbb{R}^n}\frac{4Z^2+2sZ^3}{(1+sZ)^2} = -\int_{\mathbb{R}^n}\frac{2Z^2}{1+sZ}<0$$

for all $(u,v)\in\mathcal{N}_s$. $\square$

Let us note that c_s equals $c=m_{\mathcal{N}}=m_{\mathcal{P}}$ from Lemma 3.6 in [11] and therefore corresponds to the Mountain pass level of I_s. Given Proposition 1 we are in the position to prove Theorem 1.

Proof of Theorem 1: Part (i) was proved in [11], Lemma 3.2, so let us prove (ii). First we show that the ground state energy level c_s equals c_s^*. Since we have $c_s\leq c_s^*$ by definition we have to show

$$\sup_{r>0} I_s(\sqrt{r}u,\sqrt{r}v)\geq c_s^* \qquad \text{for all } u,v\in H^1(\mathbb{R}^n) \text{ with } u,v\neq 0. \tag{9}$$

From (2) we deduce the following: if $\|u\|_{\lambda_1}^2\geq\frac{\alpha}{s}\|u\|_2^2$ then we have $I_s(\sqrt{r}u,\sqrt{r}v)\geq I_s(0,\sqrt{r}v)$ for all $v\neq 0$ and $r>0$ which implies the inequality (9). The same way one proves (9) in case $\|v\|_{\lambda_2}^2\geq\frac{\beta}{s}\|v\|_2^2$ so that it remains to prove (9) for functions (u,v) satisfying

$$\|u\|_{\lambda_1}^2<\frac{\alpha}{s}\|u\|_2^2 \qquad \text{and} \qquad \|v\|_{\lambda_2}^2<\frac{\beta}{s}\|v\|_2^2. \tag{10}$$

To this end let $r>0$ be arbitrary but fixed. From (10) we infer that the numbers

$$\begin{aligned} t(u,v) &:= \frac{\frac{\alpha}{s}\|u\|_2^2-\|u\|_{\lambda_1}^2}{\frac{\alpha}{s}\|u\|_2^2+\frac{\beta}{s}\|v\|_2^2-\|u\|_{\lambda_1}^2-\|v\|_{\lambda_2}^2}, \\ r(u,v) &:= r\cdot\Big(\frac{\alpha}{s}\|u\|_2^2+\frac{\beta}{s}\|v\|_2^2-\|u\|_{\lambda_1}^2-\|v\|_{\lambda_2}^2\Big) \end{aligned}$$

satisfy $t(u,v)\in(0,1), r(u,v)>0$ as well as

$$I_s(\sqrt{r}u,\sqrt{r}v) = -\frac{r(u,v)}{2} + \frac{1}{2s^2}\int_{\mathbb{R}^n} \ln\Big(1+\frac{r(u,v)s(\alpha u^2+\beta v^2)}{\frac{\alpha}{s}\|u\|_2^2+\frac{\beta}{s}\|v\|_2^2-\|u\|_{\lambda_1}^2-\|v\|_{\lambda_2}^2}\Big). \tag{11}$$

The concavity of ln yields

$$\begin{aligned}
&\int_{\mathbb{R}^n} \ln\Big(1+\frac{r(u,v)s(\alpha u^2+\beta v^2)}{\frac{\alpha}{s}\|u\|_2^2+\frac{\beta}{s}\|v\|_2^2-\|u\|_{\lambda_1}^2-\|v\|_{\lambda_2}^2}\Big)\\
&= \int_{\mathbb{R}^n} \ln\Big(t(u,v)\Big(1+\frac{r(u,v)s\alpha u^2}{\frac{\alpha}{s}\|u\|_2^2-\|u\|_{\lambda_1}^2}\Big) + (1-t(u,v))\Big(1+\frac{r(u,v)s\beta v^2}{\frac{\beta}{s}\|v\|_2^2-\|v\|_{\lambda_2}^2}\Big)\Big)\\
&\geq t(u,v)\int_{\mathbb{R}^n} \ln\Big(1+\frac{r(u,v)s\alpha u^2}{\frac{\alpha}{s}\|u\|_2^2-\|u\|_{\lambda_1}^2}\Big) + (1-t(u,v))\int_{\mathbb{R}^n} \ln\Big(1+\frac{r(u,v)s\beta v^2}{\frac{\beta}{s}\|v\|_2^2-\|v\|_{\lambda_2}^2}\Big)\\
&\geq \min\Big\{\int_{\mathbb{R}^n} \ln\Big(1+\frac{r(u,v)s\alpha u^2}{\frac{\alpha}{s}\|u\|_2^2-\|u\|_{\lambda_1}^2}\Big), \int_{\mathbb{R}^n} \ln\Big(1+\frac{r(u,v)s\beta v^2}{\frac{\beta}{s}\|v\|_2^2-\|v\|_{\lambda_2}^2}\Big)\Big\}.
\end{aligned}$$

Combining this inequality with (11) gives

$$\begin{aligned}
I_s(\sqrt{r}u,\sqrt{r}v) \geq \min\Big\{ &-\frac{r(u,v)}{2} + \int_{\mathbb{R}^n} \ln\Big(1+\frac{r(u,v)s\alpha u^2}{\frac{\alpha}{s}\|u\|_2^2-\|u\|_{\lambda_1}^2}\Big),\\
&-\frac{r(u,v)}{2} + \int_{\mathbb{R}^n} \ln\Big(1+\frac{r(u,v)s\beta v^2}{\frac{\beta}{s}\|v\|_2^2-\|v\|_{\lambda_2}^2}\Big)\Big\}.
\end{aligned}$$

Taking the supremum with respect to $r>0$ we get (9) and therefore $c_s \geq c_s^*$ which is what we had to show.

It remains to prove that every ground state is semitrivial unless $\lambda_1=\lambda_2, \alpha=\beta$. To this end assume that (u,v) is a fully nontrivial ground state solution of (1) so that in particular $I_s(u,v)=c_s$ holds. Then $c_s=c_s^*$ implies that the inequalities from above are equalities for some $r>0$. In particular, since ln is strictly concave and $t(u,v)\in(0,1)$ we get

$$1+\frac{r(u,v)s\alpha u^2}{\frac{\alpha}{s}\|u\|_2^2-\|u\|_{\lambda_1}^2} = k\cdot\Big(1+\frac{r(u,v)s\beta v^2}{\frac{\beta}{s}\|v\|_2^2-\|v\|_{\lambda_2}^2}\Big) \qquad \text{a.e. on } \mathbb{R}^n$$

for some $k>0$. This implies $k=1$ so that u,v have to be positive multiples of each other. From the Euler-Lagrange equation (1) we deduce $\lambda_1=\lambda_2, \alpha=\beta$ which finishes the proof. □

3. Proof of Theorem 2

In this section we assume $\lambda_1,\lambda_2,\alpha,\beta>0$ as before but the space dimension n is supposed to be $1,2$ or 3. In Remark 2 we will comment on the reason for this restriction. Let us first provide the functional analytical framework we will be working in. In case $n\geq 2$ we set $X := H_r^1(\mathbb{R}^n)\times H_r^1(\mathbb{R}^n)$ to be the product of the radially symmetric functions in $H^1(\mathbb{R}^n)$ and define $F: X\times(0,\infty)\to X$ by

$$F(u,v,s) := \begin{pmatrix} u-(-\Delta+\lambda_1)^{-1}(\alpha u Z(1+sZ)^{-1}) \\ v-(-\Delta+\lambda_2)^{-1}(\beta v Z(1+sZ)^{-1})\end{pmatrix} \quad \text{where } Z:=\alpha u^2+\beta v^2. \tag{12}$$

Hence, finding solutions of (1) is equivalent to finding zeros of F. Using the compactness of the embeddings $H^1_r(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ for $n \geq 2$ and $2 < q < \frac{2n}{n-2}$ one can check that for all s the function $F(\cdot, s)$ is a smooth compact perturbation of the identity in X so that the Krasnoselski-Rabinowitz Global Bifurcation Theorem [9], [16] is applicable. In case $n = 1$, however, this structural property is not satisfied which motivates a different choice for X. In Appendix A we show that one can define a suitable Hilbert space X of exponentially decreasing functions such that $F(\cdot, s) : X \to X$ is again a smooth compact perturbation of the identity in X. Besides this technical unconvenience the case $n = 1$ can be treated in a similar way to the case $n \in \{2, 3\}$ so that we only carry out the proofs only for the latter. Furthermore we always assume $\frac{\lambda_2}{\lambda_1} < \frac{\beta}{\alpha}$ according to the assumption of Theorem 2.

The first step in our bifurcation analysis is to investigate the linearized problems associated to the equation $F(u, v, s) = 0$ around the elements of the semitrivial solution branch $\mathcal{T}_2$. While doing this we make use of a nondegeneracy result for ground states of semilinear problems which is due to Bates and Shi [2]. Amongst other things it tells us that u_s is a nondegenerate solution of the first equation in (3), i.e. we have the following result.

Proposition 2. *The linear problem*

$$-\Delta\phi + \lambda_1\phi = \frac{3\alpha^2 u_s^2 + s\alpha^3 u_s^4}{(1 + s\alpha u_s^2)^2}\phi, \qquad \phi \in H^1_r(\mathbb{R}^n), \quad 0 < s < \frac{\alpha}{\lambda_1}$$

only admits the trivial solution $\phi = 0$.

Proof. In order to apply Theorem 5.4 (6) from [2] we set

$$g(z) := -\lambda_1 z + \frac{\alpha^2 z^3}{1 + s\alpha z^2} \qquad (z \in \mathbb{R})$$

so that u_s is the ground state solution of $-\Delta u = g(u)$ in $\mathbb{R}^n$ which is centered at the origin. In the notation of [2] one can check that g is of class (A). Indeed, the properties (g1),(g2),(g3A),(g4A),(g5A) from page 258 in [2] are satisfied for $b = (\frac{\lambda_1}{\alpha^2 - \alpha\lambda_1 s})^{1/2}, K_\infty = 1$ and the unique positive number $\theta > b$ satisfying $(\frac{\alpha}{s} - \lambda_1)\theta^2 - \frac{1}{s^2}\ln(1 + s\alpha\theta^2) = 0$. Notice that (g4A),(g5A) follow from the fact that $K_g(z) := zg'(z)/g(z)$ decreases from 1 to $-\infty$ on the interval $(0, b)$ and that it decreases from $+\infty$ to $K_\infty = 1$ on (b, ∞). Having checked the assumptions of Theorem 5.4 (6) we obtain that the space of solutions of $-\Delta\phi - g'(u_s)\phi = 0$ in $\mathbb{R}^n$ is spanned by $\partial_1 u_s, \ldots, \partial_n u_s$ implying that the linear problem only has the trivial solution in $H^1_r(\mathbb{R}^n)$. Due to

$$g'(u_s) = -\lambda_1 + \frac{3\alpha^2 u_s^2 + s\alpha^3 u_s^4}{(1 + s\alpha u_s^2)^2} \tag{13}$$

this proves the claim. $\square$

Using this preliminary result we can characterize all possible bifurcation points on $\mathcal{T}_2$ which are, due to the Implicit Function Theorem, the points where the kernel of the linearized operators are nontrivial. For notational purposes we introduce the linear compact self-adjoint

operator $L(s) : H_r^1(\mathbb{R}^n) \to H_r^1(\mathbb{R}^n)$ for parameters $0 < s < \frac{\alpha}{\lambda_1}$ by setting

$$L(s)\phi := (-\Delta + \lambda_2)^{-1}(W_s\phi), \qquad W_s(x) := \frac{\alpha\beta u_s(x)^2}{1 + s\alpha u_s(x)^2}, \qquad (0 < s < \frac{\alpha}{\lambda_1})$$

for $\phi \in H_r^1(\mathbb{R}^n)$. Denoting by $(\mu_k(s))_{k\in\mathbb{N}_0}$ the decreasing null sequence of eigenvalues of $L(s)$ we will observe that finding bifurcation points on $\mathcal{T}_2$ amounts to solving $\mu_k(s) = 1$ for $s \in (0, \frac{\alpha}{\lambda_1})$ and $k \in \mathbb{N}_0$. In fact we have the following.

Proposition 3. *We have*

$$\ker(\partial_X F(u_s, 0, s)) = \{0\} \times \ker(\mathrm{Id} - L(s)) \qquad \textit{for } 0 < s < \frac{\alpha}{\lambda_1}.$$

Proof. For $(u, v), (\phi_1, \phi_2) \in X$ we have

$$\partial_X F_1(u, v, s)[\phi_1, \phi_2] = \phi_1 - (-\Delta + \lambda_1)^{-1}\Big(\frac{s\alpha Z^2 + 3\alpha^2 u^2 + \alpha\beta v^2}{(1 + sZ)^2}\phi_1 + \frac{2\alpha\beta uv}{(1 + sZ)^2}\phi_2\Big)$$

$$\partial_X F_2(u, v, s)[\phi_1, \phi_2] = \phi_2 - (-\Delta + \lambda_2)^{-1}\Big(\frac{s\beta Z^2 + 3\beta^2 v^2 + \alpha\beta u^2}{(1 + sZ)^2}\phi_2 + \frac{2\alpha\beta uv}{(1 + sZ)^2}\phi_1\Big)$$

Plugging in $u = u_s, v = 0$ and $Z = \alpha u^2 + \beta v^2 = \alpha u_s^2$ gives

$$\begin{aligned}
\partial_X F_1(u_s, 0, s)[\phi_1, \phi_2] &= \phi_1 - (-\Delta + \lambda_1)^{-1}\Big(\frac{3\alpha^2 u_s^2 + s\alpha^3 u_s^4}{(1 + s\alpha u_s^2)^2}\phi_1\Big), \\
\partial_X F_2(u_s, 0, s)[\phi_1, \phi_2] &= \phi_2 - (-\Delta + \lambda_2)^{-1}\Big(\frac{s\beta\alpha^2 u_s^4 + \alpha\beta u_s^2}{(1 + s\alpha u_s^2)^2}\phi_2\Big) \\
&= \phi_2 - (-\Delta + \lambda_2)^{-1}\Big(\frac{\alpha\beta u_s^2}{1 + s\alpha u_s^2}\phi_2\Big) \\
&= \phi_2 - (-\Delta + \lambda_2)^{-1}(W_s\phi_2) \\
&= \phi_2 - L(s)\phi_2.
\end{aligned}$$

From these formulas and Proposition 2 we deduce the claim. □

Given this result our aim is to find sufficient conditions for the equation $\mu_k(s) = 1$ to be solvable. Since there is only few information available for any given $s > 0$ our approach consists of proving the continuity of μ_k and calculating the limits of $\mu_k(s)$ as s approaches the boundary of $(0, \frac{\alpha}{\lambda_1})$. It will turn out that the limits at both sides of the interval exist and that they lie on opposite sides of the value 1 provided our sufficient conditions from Theorem 2 are satisfied. As a consequence these conditions and the Intermediate Value Theorem imply the solvability of $\mu_k(s) = 1$ and it remains to add some technical arguments in order to apply the Krasnoselski-Rabinowitz Global Bifurcation Theorem to prove Theorem 2. Calculating the limits of μ_k at the ends of $(0, \frac{\alpha}{\lambda_1})$ requires the Propositions 4 and 5.

Proposition 4. *We have*

$$u_s \to u_0, \qquad W_s \to \alpha\beta u_0^2 \qquad \textit{as } s \to 0$$

where the convergence is uniform on $\mathbb{R}^n$.

Proof. As in Lemma 1 in Appendix A one shows that on every interval $[0, s_0]$ with $0 < s_0 < \frac{\alpha}{\lambda_1}$ there is an exponentially decreasing function which bounds each of the functions u_s with $s \in [0, s_0]$ from above. In particular, the Arzelà-Ascoli Theorem shows $u_s \to u_0$ and $W_s \to \alpha\beta u_0^2$ as $s \to 0$ locally uniformly on $\mathbb{R}^n$ so that the uniform exponential decay gives $u_s \to u_0$ and $W_s \to \alpha\beta u_0^2$ uniformly on $\mathbb{R}^n$. $\square$

Proposition 5. *We have*

$$u_s \to +\infty, \qquad W_s \to \frac{\beta\lambda_1}{\alpha} \qquad \text{as } s \to \frac{\alpha}{\lambda_1}$$

where the convergence is uniform on bounded sets in $\mathbb{R}^n$.

Proof. First we show

$$u_s(0) = \max_{\mathbb{R}^n} u_s \to \infty \qquad \text{as } s \to s^* := \frac{\alpha}{\lambda_1}. \tag{14}$$

Otherwise we would observe $u_s(0) \to a$ for some subsequence where $a \geq 0$. In case $a > 0$ a combination of elliptic regularity theory for (3) and the Arzelà-Ascoli Theorem would imply that u_s converges locally uniformly to a nontrivial radially symmetric function $u \in C^1(\mathbb{R}^n)$ satisfying $-\Delta u + \lambda_2 u = \frac{\alpha^2 u^3}{1+s^*\alpha u^2}$ in $\mathbb{R}^n$ in the weak sense and $u(0) = \|u\|_\infty = a$. As in Lemma 1 we conclude that the functions u_s are uniformly exponentially decaying so that u even lies in $H_r^1(\mathbb{R}^n)$. Hence, we may test the differential equation with u and obtain

$$\lambda_1 \int_{\mathbb{R}^n} u^2 \leq \int_{\mathbb{R}^n} |\nabla u|^2 + \lambda_1 u^2 = \int_{\mathbb{R}^n} \frac{\alpha^2 u^4}{1 + s^*\alpha u^2} < \frac{\alpha}{s^*} \int_{\mathbb{R}^n} u^2 = \lambda_1 \int_{\mathbb{R}^n} u^2$$

which is impossible. It therefore remains to exclude the case $a = 0$. In this case the functions u_s would converge uniformly on $\mathbb{R}^n$ to the trivial solution implying that $u_s/u_s(0)$ would converge to a nonnegative bounded function $\phi \in C^1(\mathbb{R}^n)$ satisfying $-\Delta\phi + \lambda_1\phi = 0$ on $\mathbb{R}^n$ and $\phi(0) = \|\phi\|_\infty = 1$. Hence, ϕ is smooth so that Liouville's Theorem applied to the function $(x, y) \mapsto \phi(x)\cos(\sqrt{\lambda_1}y)$ defined on $\mathbb{R}^{n+1}$ implies that ϕ is constant and thus $\phi \equiv 0$ contradicting $\phi(0) = 1$. This proves (14).

Now set $\phi_s := u_s/u_s(0)$. Using

$$-\Delta\phi_s + \lambda_1\phi_s = \alpha\phi_s \cdot \frac{\alpha u_s^2}{1 + s\alpha u_s^2} \qquad \text{in } \mathbb{R}^n$$

and the fact that $\alpha u_s^2/(1 + s\alpha u_s^2)$ remains bounded as $s \to s^*$ we get that the functions ϕ_s converge locally uniformly as $s \to s^*$ to some nonnegative radially nonincreasing function $\phi \in C^1(\mathbb{R}^n)$ satisfying $\phi(0) = \|\phi\|_\infty = 1$. In order to prove our claim it is sufficient to show $\phi \equiv 1$ since this implies $u_s = u_s(0)\phi_s \to \infty$ locally uniformly and in particular $W_s \to \frac{\beta\lambda_1}{\alpha}$ locally uniformly.

First we show $\phi > 0$. If this were not true then there would exist a smallest number $\rho \in (0, \infty)$ such that $\phi|_{B_\rho} > 0$ and $\phi|_{\partial B_r} = 0$ for all $r \in [\rho, \infty)$. On B_ρ we have $v_s \to \infty$ and $\alpha^2 u_s^2/(1 + s\alpha u_s^2) \to \lambda_1$ implies $-\Delta\phi + \lambda_1\phi = \lambda_1\phi$ in B_ρ and $\phi|_{\partial B_\rho} = 0$ in contradiction to the maximum principle. Hence, we must have $\phi > 0$ in $\mathbb{R}^n$. Repeating the above argument

we find $-\Delta\phi + \lambda_1\phi = \lambda_1\phi$ in $\mathbb{R}^n$ and $\phi(0) = \|\phi\|_\infty = 1$ so that Liouville's Theorem implies $\phi \equiv \phi(0) = 1$. □

The previous Proposition enables us to calculate the limits of the eigenvalue functions $\mu_k(s)$ as s approaches the boundary of $(0, \frac{\alpha}{\lambda_1})$.

Proposition 6. *For all $k \in \mathbb{N}_0$ the functions μ_k are positive and continuous on $(0, \frac{\alpha}{\lambda_1})$. Moreover we have*

$$\mu_k(s) \to \bar{\mu}_k \quad \text{as } s \to 0, \qquad \mu_k(s) \to \frac{\beta\lambda_1}{\alpha\lambda_2} \quad \text{as } s \to \frac{\alpha}{\lambda_1}$$

Proof. As in Proposition 4 the uniform exponential decay of the functions u_s for $s \in [0, s^*)$ for $s^* := \frac{\alpha}{\lambda_1}$ implies $u_s \to u_{s_0}, W_s \to W_{s_0}$ uniformly on $\mathbb{R}^n$ whenever $s_0 \in [0, s^*]$. Hence, the Courant-Fischer min-max-characterization for the eigenvalues $\mu_k(s)$ implies the continuity of μ_k as well as $\mu_k(s) \to \bar{\mu}_k$ as $s \to 0$.

In order to evaluate $\mu_k(s)$ for $s \to s^*$ we apply Lemma 1 from Appendix C. The conditions (i) and (ii) of the Lemma are satisfied since we have $\|W_s\|_\infty = W_s(0) \to \frac{\beta\lambda_1}{\alpha}$ and $W_s \to \frac{\beta\lambda_1}{\alpha}$ locally uniformly as $s \to s^*$ by Proposition 5. From the Lemma we get $\mu_k(s) \to \frac{\beta\lambda_1}{\alpha\lambda_2}$ as $s \to s^*$ which is all we had to show. □

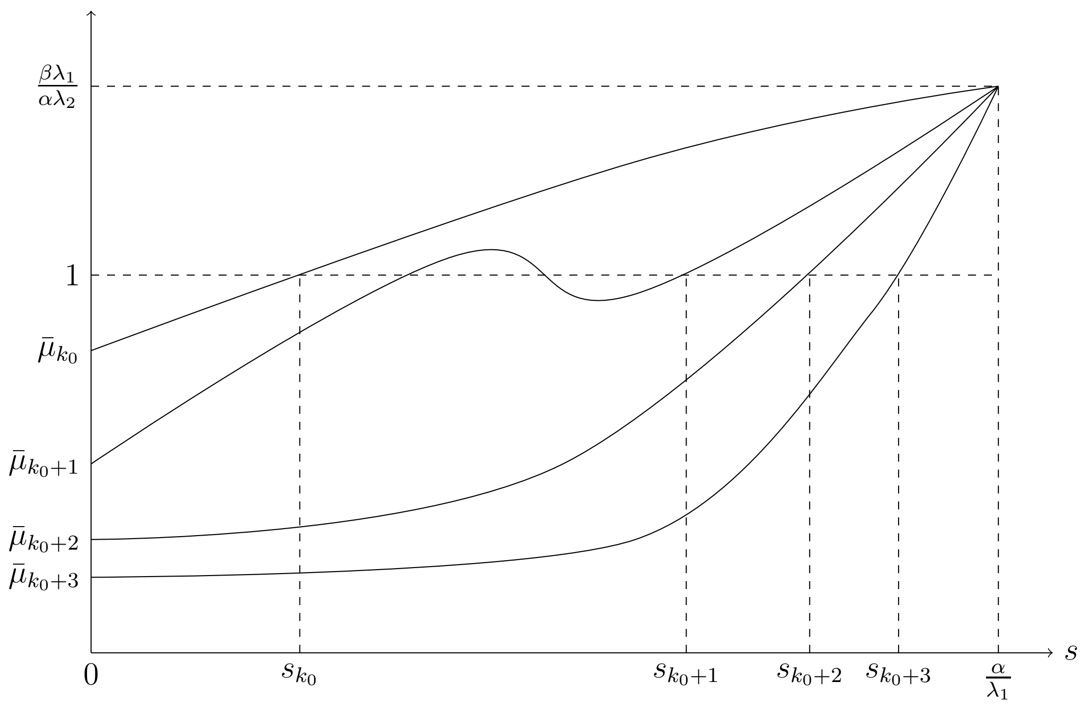

FIGURE 1. The eigenvalue functions $\mu_{k_0}, \ldots, \mu_{k_0+3}$ on $(0, \frac{\alpha}{\lambda_1})$.

Remark 2. *When $n \geq 4$ the statement of Proposition 4 is not meaningful since u_0 does not exist in this case by Pohožaev's identity. So it is natural to ask how u_s, W_s and μ_k behave when s approaches zero and $n \geq 4$. Having found an answer to this question it might be possible to modify our reasoning in order to prove sufficient conditions for the existence of bifurcation points from $\mathcal{T}_2$ in the case $n \geq 4$.*

The above Propositions are sufficient for proving the mere existence of the continua $\mathcal{C}_k$ from Theorem 2. So it remains to show that positive solutions lie to the left of the threshold value $\frac{\alpha-\beta}{\lambda_1-\lambda_2}$ and that they are equibounded in X. The latter result will be proved in Lemma 1 whereas the first claim follows from the following nonexistence result which slightly improves Theorem 3.10 and Theorem 3.11 from [11].

Proposition 7. *If positive solutions of* (1) *exist then we either have*

$$\textit{(i)}\quad \lambda_1 = \lambda_2, \alpha = \beta \qquad\qquad \textit{or} \qquad\qquad \textit{(ii)}\quad s < \frac{\alpha-\beta}{\lambda_1-\lambda_2} < \min\Big\{\frac{\alpha}{\lambda_1}, \frac{\beta}{\lambda_2}\Big\}.$$

Proof. Assume there is a positive solution (u,v) of (1). Testing (1) with (v,u) leads to

$$\int_{\mathbb{R}^n} uv\Big(\lambda_1 - \lambda_2 - (\alpha-\beta)\frac{Z}{1+sZ}\Big) = 0.$$

Hence, the function $\lambda_1 - \lambda_2 - (\alpha-\beta)\frac{Z}{1+sZ}$ vanishes identically or it changes sign in $\mathbb{R}^n$. In the first case we get (i), so let us assume that the function changes sign. Then we have $\lambda_1 \neq \lambda_2$ and $\alpha \neq \beta$ so that Theorem 3.11 and Remark 3.18 in [11] imply $0 < \frac{\alpha-\beta}{\lambda_1-\lambda_2} < \min\{\frac{\alpha}{\lambda_1}, \frac{\beta}{\lambda_2}\}$. Moreover, $s \geq \frac{\alpha-\beta}{\lambda_1-\lambda_2}$ would imply

$$\Big|\lambda_1 - \lambda_2 - (\alpha-\beta)\frac{Z}{1+sZ}\Big| > |\lambda_1-\lambda_2| - \frac{|\alpha-\beta|}{s} \geq 0 \qquad \text{on } \mathbb{R}^n$$

contradicting the assumption that $\lambda_1 - \lambda_2 - (\alpha-\beta)\frac{Z}{1+sZ}$ changes sign. Hence, we have $s < \frac{\alpha-\beta}{\lambda_1-\lambda_2}$ which concludes the proof. ☐

Proof of Theorem 2: The main ingredient of our proof is the Krasnoselski-Rabinowitz Global Bifurcation Theorem (cf. [9], [16] or [8], Theorem II.3.3), which, roughly speaking, says that a change of the Leray-Schauder index along a given solution curve over some parameter interval implies the existence of a bifurcating continuum emanating from the solution curve within this parameter interval. In our application the solution curve is $\mathcal{T}_2$ and the first task is to identify parameter intervals within $(0, \frac{\alpha}{\lambda_1})$ where the index changes. For notational purposes we set $s^* := \frac{\alpha}{\lambda_1}$.

1st step: Existence of solution continua $\mathcal{C}_k$ bifurcating from $\mathcal{T}_2$. By assumption of the Theorem and Proposition 6 we have

$$\lim_{s\to 0} \mu_k(s) = \bar{\mu}_k < 1, \qquad \lim_{s\to s^*} \mu_k(s) = \frac{\beta\lambda_1}{\alpha\lambda_2} > 1 \qquad \text{for all } k \geq k_0.$$

The continuity of the μ_k on $(0,s^*)$ as well as $\mu_k(s)>\mu_{k+1}(s)$ for all $k\geq k_0, s\in(0,s^*)$ therefore implies $0<a_{k_0}<a_{k_0+1}<a_{k_0+2}<\ldots<\frac{\alpha}{\lambda_1}$ for the numbers a_k given by

$$a_k:=\sup\Big\{0<s<\frac{\alpha}{\lambda_1}:\mu_k(s)<1\Big\}\qquad (k\geq k_0).$$

By definition of a_k we can find $\underline{a}_k<a_k<\overline{a}_k$ such that the following holds:

$$\begin{aligned}
&\text{(i)} && \mu_k(s)<1<\mu_{k-1}(\underline{a}_k) && \text{for all } s\leq\underline{a}_k, k\geq k_0,\\
&\text{(ii)} && \mu_k(s)>1>\mu_{k-1}(\overline{a}_k) && \text{for all } s\geq\overline{a}_k, k\geq k_0,\\
&\text{(iii)} && a_k-\frac{1}{k}<\underline{a}_k<\overline{a}_k<\underline{a}_{k+1} && \text{for all } k\geq k_0.
\end{aligned}\tag{15}$$

In fact one first chooses $\overline{a}_k\in(a_k,a_{k+1})$ such that (ii) is satisfied and then $\underline{a}_k<a_k$ sufficiently close to a_k such that (i) and (iii) hold. Now let us show that the Leray-Schauder index $\mathrm{ind}(F(\cdot,s),(u_s,0))$ changes sign on each of the mutually disjoint intervals $(\underline{a}_k,\overline{a}_k)$.

The index of $F(\cdot,s)$ near $(u_s,0)$ is computed using the Leray-Schauder formula which involves the algebraic multiplicities of the eigenvalues $\mu>1$ of the compact linear operator $\mathrm{Id}-\partial_X F(u_s,0,s)$, see (II.2.11) in [8]. From the formulas appearing in Proposition 3 we find that $\mu>1$ is such an eigenvalue if and only if one of the following equations is solvable:

$$\begin{aligned}
(-\Delta+\lambda_1)^{-1}\Big(\frac{3\alpha^2u_s^2+s\alpha^3u_s^4}{(1+s\alpha u_s^2)^2}\phi\Big)&=\mu\phi\quad\text{in }\mathbb{R}^n,\quad \phi\in H_r^1(\mathbb{R}^n),\phi\neq0,\\
L(s)\psi=(-\Delta+\lambda_2)^{-1}(W_s\psi)&=\mu\psi\quad\text{in }\mathbb{R}^n,\quad \psi\in H_r^1(\mathbb{R}^n),\psi\neq0.
\end{aligned}$$

When $s=\underline{a}_k$ then the second equation is solvable with $\mu>1$ if and only if μ is an eigenvalue of $L(\underline{a}_k)$ larger than 1. By (15) (i) this is equivalent to $\mu\in\{\mu_0(\underline{a}_k),\ldots,\mu_{k-1}(\underline{a}_k)\}$. Due to Sturm-Liouville theory each of these eigenvalues is simple. The first equation is solvable with $\mu>1$ if and only if $\Delta+g'(u_s)$ has a negative eigenvalue in $H_r^1(\mathbb{R}^n)$ where g is defined as in (13). From Theorem 5.4 (4)-(6) in [2] we infer that there is precisely one such eigenvalue $\mu>1$ and μ has algebraic multiplicity one. Denoting the $H_r^1(\mathbb{R}^n)$-spectrum with σ we arrive at the formula

$$\begin{aligned}
\mathrm{ind}(F(\cdot,\underline{a}_k),(0,v_{\underline{a}_k}))&=(-1)^{\#\{\mu\in\sigma(\mathrm{Id}-\partial_XF(0,v_{\underline{a}_k},\underline{a}_k)):\mu>1\}}\\
&=(-1)^{k+1}\\
&=-(-1)^{k+2}\\
&=-(-1)^{\#\{\mu\in\sigma(\mathrm{Id}-\partial_XF(0,v_{\overline{a}_k},\overline{a}_k)):\mu>1\}}\\
&=-\mathrm{ind}(F(\cdot,\overline{a}_k),(0,v_{\overline{a}_k})).
\end{aligned}$$

The Krasnoselski-Rabinowitz Theorem implies that the interval $(\underline{a}_k,\overline{a}_k)$ contains a least one bifurcation point $(u_{s_k},0,s_k)$ so that the maximal component $\mathcal{C}_k$ in $\mathcal{S}$ satisfying $(u_{s_k},0,s_k)\in\mathcal{C}_k$ is non-void. By Proposition 3 this implies $\mu_j(s_k)=1$ for some $j\in\mathbb{N}_0$ and (15) implies $j=k$, i.e. $\mu_k(s_k)=1$. Indeed, property (ii) gives $\mu_{k-1}(s_k)>1$ and (i) gives $\mu_{k+1}(s_k)<1$.

2nd step: $s_k\to s^*$ as $k\to\infty$. If not then we would have $s_k\to\bar{s}$ from below for some $\bar{s}<s^*$. From $s_k\in(\underline{a}_k,\overline{a}_k)$, the inequality $\underline{a}_k>a_k-1/k$ and the definition of a_k we deduce

$\mu_k(t) \geq 1$ whenever $t \geq s_k + \frac{1}{k}, k \geq k_0$ and thus

$$\mu_k(t) \geq 1 \qquad \text{for all } t \in \Big(\frac{\bar{s}+s^*}{2}, s^*\Big) \text{ and } k \geq k_1$$

for some sufficiently large $k_1 \in \mathbb{N}$. This contradicts $\mu_k(t) \to 0$ as $k \to \infty$ for all $t \in (0, s^*)$ and the claim is proved.

3nd step: Existence of seminodal solutions within $\mathcal{C}_k$. We briefly show that fully nontrivial solutions of (1) belonging to a sufficiently small neighbourhood of $(u_{s_k}, 0, s_k)$ are $(0,k)$-nodal. Indeed, if solutions (u^m, v^m, s^m) of (1) converge to $(u_{s_k}, 0, s_k)$ then $v^m/v^m(0)$ converges to the eigenfunction ϕ of $L(s_k)$ with $\phi(0) = 1$ which is associated to the eigenvalue 1. Due to $\mu_k(s_k) = 1$ and Sturm-Liouville theory ϕ has precisely $k+1$ nodal annuli so that the same is true for v^m and sufficiently large $m \in \mathbb{N}$. On other hand $u^m \to u$ implies that u^m must be positive for large m which proves the claim.

4rd step: Positive solutions. The claim concerning positive solutions of (1) follows directly from Proposition 7 and Lemma 1 from Appendix A. $\square$

4. Proof of Corollary 1 and Corollary 2

Let $\zeta \in H_r^1(\mathbb{R}^n)$ be the unique positive function which satisfies $-\Delta\zeta + \zeta = \zeta^3$ in $\mathbb{R}^n$ so that u_0, v_0 can be rewritten as

$$u_0(x) = \sqrt{\lambda_1}\alpha^{-1}\zeta(\sqrt{\lambda_1}x), \qquad v_0(x) = \sqrt{\lambda_2}\beta^{-1}\zeta(\sqrt{\lambda_2}x).$$

Hence, Corollary 1 follows from Theorem 2 and the estimate

$$\bar{\mu}_0 = \max_{\phi\neq 0} \frac{\alpha\beta\|u_0\phi\|_2^2}{\|\phi\|_{\lambda_2}^2} \leq \max_{\phi\neq 0} \frac{\alpha\beta\|u_0\|_4^2\|\phi\|_4^2}{\|\phi\|_{\lambda_2}^2} = \frac{\alpha\beta\|u_0\|_4^2\|v_0\|_4^2}{\|v_0\|_{\lambda_2}^2} = \frac{\alpha}{\beta}\frac{\|u_0\|_4^2}{\|v_0\|_4^2} = \frac{\beta}{\alpha}\Big(\frac{\lambda_1}{\lambda_2}\Big)^{\frac{4-n}{4}}.$$

In case $n = 1$ we have $\zeta(x) = \sqrt{2}\,\mathrm{sech}(x)$ and it is known (see for instance Lemma 5.1 in [4]) that the eigenvalue problem $\mu(-\phi'' + \omega^2\phi) = \zeta^2\phi$ in $\mathbb{R}$ admits nontrivial solutions in $H_r^1(\mathbb{R})$ if and only if $2/\mu = (\omega + 2k)(\omega + 2k + 1)$ for some $k \in \mathbb{N}_0$. This implies

$$\bar{\mu}_k = \frac{\beta}{\alpha}\frac{2}{(\sqrt{\frac{\lambda_2}{\lambda_1}} + 2k)(\sqrt{\frac{\lambda_2}{\lambda_1}} + 2k + 1)} \qquad (k \in \mathbb{N}_0)$$

and Corollary 2 follows from Theorem 2.

5. Open problems

Let us finally summarize some open problems concerning (1) which we were not able to solve and which we believe to provide a better understanding of the equation. Especially the open questions concerning global bifurcation scenarios are supposed to be very difficult from the analytical point of view so that numerical indications would be very helpful, too. The following questions might be of interest:

(1) As in the author's work on weakly coupled nonlinear Schrödinger systems [12] one could try to prove the existence of positive solutions by minimizing the Euler functional over the "system Nehari manifold" $\mathcal{M}_s$ consisting of all fully nontrivial functions $(u,v) \in X$ which satisfy $I'(u,v)[(u,0)] = I'(u,v)[(0,v)] = 0$. For which parameter values $\alpha, \beta, \lambda_1, \lambda_2, s$ are there such minimizers and do they belong to $\mathcal{C}_0$?
(2) What is the existence theory and the bifurcation scenario when $\alpha\lambda_2 = \beta\lambda_1$ and $\alpha \neq \beta, \lambda_1 \neq \lambda_2$?
(3) In the case $\alpha = \beta, \lambda_1 = \lambda_2$ the points on $\mathcal{T}_1, \mathcal{T}_2$ are connected by a smooth curve and the same is true for every semitrivial solution. Do these connections break up when the parameters of the equation are perturbed? This is related to the question whether the continuum $\mathcal{C}_0$ contains $\mathcal{T}_1$.
(4) It would be interesting to know if the eigenvalue functions μ_k are strictly monotone. The monotonicity of μ_k would imply that s_k are the only solutions of $\mu_k(s) = 1$ so that the totality of bifurcation points is given by $(s_k)_{k \geq k_0}$.
(5) We expect that $\mathcal{T}_1, \mathcal{T}_2$ extend to semitrivial solution branches $\tilde{\mathcal{T}}_1, \tilde{\mathcal{T}}_2$ containing also negative parameter values s. A bifurcation analysis for such branches remains open. Let us shortly comment why we expect an interesting outcome of such a study. In the model case $n = 1$ and $\beta = \lambda_2 = 1$ one obtains from (4) the existence of u_s for all $s < 0$ as well as the enclosure $u_s(0)^2 \in (1/(|s|+1), 1/|s|)$. Using this one successively proves $su_s(0)^2 \to -1$ and $s(1 + su_s(0)^2) \to 0$ as $s \to -\infty$. This implies $W_s(0) = u_s(0)^2/(1+su_s(0)^2) \to +\infty$ as $s \to -\infty$ so that one expects $\mu_k(s) \to +\infty$ as $s \to -\infty$ for all $k \in \mathbb{N}_0$. In view of $\bar{\mu}_{k_0} < 1$ this leads to the natural conjecture that there are infinitely many bifurcating branches $(\tilde{\mathcal{C}}_k)_{k \geq k_0}$ also in the parameter range $s < 0$.
(6) Our paper does not contain any existence result for fully nontrivial solutions when $n \geq 4$ and $\lambda_1 \neq \lambda_2$ or $\alpha \neq \beta$. It would be interesting to know whether there is a nonexistence result behind or whether an analysis following Remark 2 yields existence of fully nontrivial solutions.

6. Appendix A

In our proof of the a priori bounds for positive solutions (u,v) of (1) we will use the notation $s^* := \min\{\frac{\alpha}{\lambda_1}, \frac{\beta}{\lambda_2}\}$ and $u(x) = \hat{u}(|x|), v(x) = \hat{v}(|x|)$ so that $\hat{u}, \hat{v}$ denote the radial profiles of u, v. Notice that all nonnegative solutions are radially symmetric and radially decreasing by Lemma 3.8 in [11]. We want to highlight the fact that the main ideas leading to Lemma 1 are taken from section 2 in [7].

Lemma 1. *Let $n \in \{1,2,3\}$. For all $\varepsilon > 0$ there are $c_\varepsilon, C_\varepsilon > 0$ such that all nonnegative solutions (u,v) of (1) for $\lambda_1, \lambda_2, \alpha, \beta \in [\varepsilon, \varepsilon^{-1}]$ and $s \in [0, \min\{\frac{\alpha}{\lambda_1}, \frac{\beta}{\lambda_2}\} - \varepsilon]$ satisfy*

$$\|u\|_{\lambda_1} + \|v\|_{\lambda_2} < C_\varepsilon, \qquad u(x) + v(x) \leq C_\varepsilon e^{-c_\varepsilon |x|} \quad \textit{for all } x \in \mathbb{R}^n.$$

Proof. 1st step: Boundedness in $L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$. Assume that there is an sequence (u_k, v_k) of nonnegative solutions of (1) for parameters $(\lambda_1)_k, (\lambda_2)_k, \alpha_k, \beta_k \in [\varepsilon, \varepsilon^{-1}]$ and

$s_k \in [0, s^* - \varepsilon]$ which is unbounded in $L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$. As always we write $Z_k(x) := \alpha_k u_k(x)^2 + \beta_k v_k(x)^2$. Passing to a subsequence we may assume $Z_k(0) = \max_{\mathbb{R}^n} Z_k \to \infty$ and $((\lambda_1)_k, (\lambda_2)_k, \alpha_k, \beta_k, s_k) \to (\lambda_1, \lambda_2, \alpha, \beta, s)$ for some $s \in [0, s^* - \varepsilon]$ and $\lambda_1, \lambda_2, \alpha, \beta \in [\varepsilon, \varepsilon^{-1}]$. Let us distinguish the cases $s > 0$ and $s = 0$ to lead this assumption to a contradiction.

The case $s > 0$. The functions

$$\phi_k := u_k Z_k(0)^{-1/2}, \qquad \psi_k := v_k Z_k(0)^{-1/2}$$

are bounded in $L^\infty(\mathbb{R}^n)$ and satisfy $\alpha_k \phi_k(0)^2 + \beta_k \psi_k(0)^2 = 1$ as well as

$$\begin{aligned} -\Delta\phi_k + (\lambda_1)_k \phi_k &= \alpha_k \phi_k \cdot \frac{Z_k}{1 + s_k Z_k} \quad \text{in } \mathbb{R}^n, \\ -\Delta\psi_k + (\lambda_2)_k \phi_k &= \beta_k \psi_k \cdot \frac{Z_k}{1 + s_k Z_k} \quad \text{in } \mathbb{R}^n. \end{aligned}$$

Using $Z_k/(1 + s_k Z_k) \leq s_k^{-1} = s^{-1} + o(1)$ and deGiorgi-Nash-Moser estimates we obtain from the Arzelà-Ascoli Theorem that there are bounded nonnegative radially symmetric limit functions $\phi, \psi \in C^1(\mathbb{R}^n)$ satisfying $\alpha\phi(0)^2 + \beta\psi(0)^2 = 1$ and

$$-\Delta\phi + \lambda_1 \phi = \frac{\alpha}{s}\phi \quad \text{in } \mathbb{R}^n, \qquad -\Delta\psi + \lambda_2 \psi = \frac{\beta}{s}\psi \quad \text{in } \mathbb{R}^n.$$

From $\lambda_1 < \frac{\alpha}{s}$ and $\lambda_2 < \frac{\beta}{s}$ we obtain

$$\phi(r) = \kappa_1 r^{\frac{2-n}{2}} J_{\frac{n-2}{2}}\Big(\Big(\frac{\alpha}{s} - \lambda_1\Big)^{1/2} r\Big), \qquad \psi(r) = \kappa_2 r^{\frac{2-n}{2}} J_{\frac{n-2}{2}}\Big(\Big(\frac{\beta}{s} - \lambda_2\Big)^{1/2} r\Big) \quad \text{for } r \geq 0$$

for some $\kappa_1, \kappa_2 \in \mathbb{R}$. Since the functions ϕ, ψ are nonnegative this is only possible in case $\kappa_1 = \kappa_2 = 0$ which contradicts $\alpha\phi(0)^2 + \beta\psi(0)^2 = 1$. Hence the case $s > 0$ does not occur.

The case $s = 0$. We first show $s_k Z_k \to 0$ uniformly on $\mathbb{R}^n$ which, due to $Z_k(0) = \max_{\mathbb{R}^n} Z_k$, is equivalent to proving $s_k Z_k(0) \to 0$. So let κ be an arbitrary accumulation point of the sequence $(s_k Z_k(0))_{k \in \mathbb{N}}$ and without loss of generality we assume $s_k Z_k(0) \to \kappa \in [0, \infty]$ so that we are left to show $\kappa = 0$. To this end set

$$\phi_k(x) := u_k(\sqrt{s_k}x) Z_k(0)^{-1/2}, \qquad \psi_k(x) := v_k(\sqrt{s_k}x) Z_k(0)^{-1/2}.$$

The functions ϕ_k, ψ_k satisfy $\alpha_k \phi_k(0)^2 + \beta_k \psi_k(0)^2 = 1$ as well as

$$\begin{aligned} -\Delta\phi_k + s_k(\lambda_1)_k \phi_k &= \alpha_k \phi_k \cdot \frac{s_k Z_k}{1 + s_k Z_k} = \alpha_k \phi_k \cdot \frac{s_k Z_k(0)(\alpha_k \phi_k^2 + \beta_k \psi_k^2)}{1 + s_k Z_k(0)(\alpha_k \phi_k^2 + \beta_k \psi_k^2)} \quad \text{in } \mathbb{R}^n, \\ -\Delta\psi_k + s_k(\lambda_2)_k \psi_k &= \beta_k \psi_k \cdot \frac{s_k Z_k}{1 + s_k Z_k} = \beta_k \psi_k \cdot \frac{s_k Z_k(0)(\alpha_k \phi_k^2 + \beta_k \psi_k^2)}{1 + s_k Z_k(0)(\alpha_k \phi_k^2 + \beta_k \psi_k^2)} \quad \text{in } \mathbb{R}^n. \end{aligned}$$

The Arzelà-Ascoli Theorem implies that a subsequence $(\phi_k), (\psi_k)$ converges locally uniformly to nonnegative functions $\phi, \psi \in C^1(\mathbb{R}^n)$ satisfying $\alpha\phi(0)^2 + \beta\psi(0)^2 = 1$ and

$$\begin{aligned} -\Delta\phi &= \alpha\phi \cdot \frac{\kappa(\alpha\phi^2 + \beta\psi^2)}{1 + \kappa(\alpha\phi^2 + \beta\psi^2)} \quad \text{in } \mathbb{R}^n, \\ -\Delta\psi &= \beta\psi \cdot \frac{\kappa(\alpha\phi^2 + \beta\psi^2)}{1 + \kappa(\alpha\phi^2 + \beta\psi^2)} \quad \text{in } \mathbb{R}^n. \end{aligned}$$

In case $\kappa = +\infty$ we arive at a contradiction as in the case $s > 0$ so let us assume $\kappa < \infty$. Then $z := \phi + \psi$ is nonnegative, nontrivial and the inequality $\alpha\phi^2 + \beta\psi^2 \leq \alpha\phi(0)^2 + \beta\psi(0)^2 = 1$ implies

$$\begin{aligned}
-\Delta z &= (\alpha\phi + \beta\psi) \cdot \frac{\kappa(\alpha\phi^2 + \beta\psi^2)}{1 + \kappa(\alpha\phi^2 + \beta\psi^2)} \\
&\geq \min\{\alpha, \beta\}(\phi + \psi) \cdot \frac{\kappa}{1 + \kappa}(\alpha\phi^2 + \beta\psi^2) \\
&\geq c(\kappa)(\phi + \psi)^3 \\
&= c(\kappa) z^3
\end{aligned}$$

where $c(\kappa) = \min\{\alpha, \beta\}^2 \kappa/(2(1 + \kappa))$. From Theorem 8.4 in [15] we infer $c(\kappa) = 0$ and thus $\kappa = 0$. Hence, every accumulation point of the sequence $(s_k Z_k(0))$ is zero so that $s_k Z_k$ converges to the trivial function uniformly on $\mathbb{R}^n$.

With this result at hand one can use the classical blow-up technique by considering

$$\tilde{\phi}_k(x) := u_k(Z_k(0)^{-1/2}x)Z_k(0)^{-1/2}, \qquad \tilde{\psi}_k(x) := v_k(Z_k(0)^{-1/2}x)Z_k(0)^{-1/2}.$$

These functions satisfy $\alpha_k\tilde{\phi}_k(0)^2 + \beta_k\tilde{\psi}_k(0)^2 = 1$ as well as

$$\begin{aligned}
-\Delta\tilde{\phi}_k + Z_k(0)^{-1}(\lambda_1)_k\tilde{\phi}_k &= \alpha_k\tilde{\phi}_k \cdot \frac{Z_k Z_k(0)^{-1}}{1 + s_k Z_k} \qquad \text{in } \mathbb{R}^n, \\
-\Delta\tilde{\psi}_k + Z_k(0)^{-1}(\lambda_2)_k\tilde{\psi}_k &= \beta_k\tilde{\psi}_k \cdot \frac{Z_k Z_k(0)^{-1}}{1 + s_k Z_k} \qquad \text{in } \mathbb{R}^n.
\end{aligned}$$

Then $s_k Z_k \to 0$ uniformly in $\mathbb{R}^n$ and similar arguments as the ones used above lead to a bounded nonnegative nontrivial solution ϕ, ψ of

$$\begin{aligned}
-\Delta\phi &= \alpha\phi(\alpha\phi^2 + \beta\psi^2) \qquad \text{in } \mathbb{R}^n, \\
-\Delta\psi &= \beta\psi(\alpha\phi^2 + \beta\psi^2) \qquad \text{in } \mathbb{R}^n,
\end{aligned}$$

which we may lead to a contradiction as above. This finally shows that $Z_k(0) \to \infty$ is impossible also in case $s = 0$ so that the nonnegative solutions (u, v) of (1) are pointwise bounded by some constant depending on ε.

2nd step: Uniform exponential decay. Let us assume for contradiction that there is a sequence (u_k, v_k, s_k) of positive solutions of (1) satisfying

$$\hat{u}_k(r_k) + \hat{v}_k(r_k) \geq k e^{-r_k/k} \quad \text{for all } k \in \mathbb{N} \text{ and some } r_k > 0. \tag{16}$$

Due to the L^∞-bounds for (u_k, v_k) which we proved in the first step we can use deGiorgi-Nash-Moser estimates and the Arzelà-Ascoli theorem to obtain a smooth bounded radially symmetric limit function (u, v) of a suitable subsequence of (u_k, v_k). As a limit of positive radially decreasing functions u, v are also nonnegative and radially nonincreasing, in particular we may define

$$u_\infty := \lim_{r\to\infty} \hat{u}(r) \geq 0, \quad v_\infty := \lim_{r\to\infty} \hat{v}(r) \geq 0.$$

Our first aim is to show $u_\infty = v_\infty = 0$. Since $(\hat{u}, \hat{v})$ decreases to some limit at infinity we have $\hat{u}'(r), \hat{v}'(r), \hat{u}''(r), \hat{v}''(r) \to 0$ as $r \to \infty$ so that (1) implies

$$\lambda_1 u_\infty = \frac{\alpha u_\infty Z_\infty}{1 + sZ_\infty}, \quad \lambda_2 v_\infty = \frac{\beta v_\infty Z_\infty}{1 + sZ_\infty} \qquad \text{where } Z_\infty = \alpha u_\infty^2 + \beta v_\infty^2. \tag{17}$$

Now define

$$\begin{aligned} E_k(r) &:= \hat{u}_k'(r)^2 + \hat{v}_k'(r)^2 - \lambda_1 \hat{u}_k(r)^2 - \lambda_2 \hat{v}_k(r)^2 + s^{-2} g(sZ_k(r)), \\ E(r) &:= \hat{u}'(r)^2 + \hat{v}'(r)^2 - \lambda_1 \hat{u}(r)^2 - \lambda_2 \hat{v}(r)^2 + s^{-2} g(sZ(r)). \end{aligned}$$

The differential equation implies $E_k'(r) = -\frac{2(n-1)}{r}(\hat{u}_k'(r)^2 + \hat{v}_k'(r)^2) \leq 0$ so that E_k decreases to some limit at infinity. The monotonicity of $\hat{u}_k, \hat{v}_k$ and $\hat{u}_k(r), \hat{v}_k(r) \to 0$ as $r \to \infty$ imply that this limit must be 0. In particular we obtain $E_k \geq 0$ and the pointwise convergence $E_k \to E$ implies that E is a nonnegative nonincreasing function. From this we obtain

$$\begin{aligned} 0 \leq \lim_{r \to \infty} E(r) \; &= \; -\lambda_1 u_\infty^2 - \lambda_2 v_\infty^2 + s^{-2} g(sZ_\infty) \\ \overset{(17)}{=} -\frac{Z_\infty^2}{1 + sZ_\infty} + s^{-2} g(sZ_\infty) \; &= \; \frac{1}{s^2}\Big(\frac{sZ_\infty}{1 + sZ_\infty} - \ln(1 + sZ_\infty)\Big). \end{aligned}$$

This equation implies $Z_\infty = 0$ and hence $u_\infty = v_\infty = 0$.

Now let μ satisfy $0 < \mu < \sqrt{\min\{\lambda_1, \lambda_2\}}$ and choose $\delta > 0$. Due to $u_\infty = v_\infty = 0$ we may choose $r_0 > 0$ such that $\hat{u}(r_0) + \hat{v}(r_0) < \delta/2$ holds. From $\hat{u}_k(r_0) \to \hat{u}(r_0), \hat{v}_k(r_0) \to \hat{v}(r_0)$ and the fact that $\hat{u}_k, \hat{v}_k$ are decreasing we obtain $\hat{u}_k(r) + \hat{v}_k(r) \leq \delta$ for all $r \geq r_0$ and all $k \geq k_0$ for some sufficiently large $k_0 \in \mathbb{N}$. Having chosen $\delta > 0$ sufficiently small the inequality $\hat{u}_k', \hat{v}_k' \leq 0$ implies

$$-(\hat{u}_k + \hat{v}_k)'' + \mu^2(\hat{u}_k + \hat{v}_k) \leq 0 \qquad \text{on } [r_0, \infty) \text{ for all } k \geq k_0.$$

Hence, the maximum principle implies that for any given $R > r_0$ the function $w_R(r) := e^{-\mu(r - r_0)} + e^{-\mu(R - r)}$ satisfies $\hat{u}_k + \hat{v}_k \leq w_R$ on (r_0, R). Indeed, w_R dominates $\hat{u}_k + \hat{v}_k$ on the boundary of (r_0, R) due to

$$w_R(r_0) = w_R(R) \geq 1 \geq \delta \geq (\hat{u}_k + \hat{v}_k)(r_0) = \max\{(\hat{u}_k + \hat{v}_k)(r_0), (\hat{u}_k + \hat{v}_k)(R)\}.$$

Sending R to infinity we obtain

$$(\hat{u}_k + \hat{v}_k)(r) \leq e^{-\mu(r - r_0)} \quad \text{for all } r \geq r_0.$$

which, together with the a priori bounds from the first step, yields a contradiction to the assumption (16). This proves the uniform exponential decay.

3rd step: Conclusion. Given the uniform exponential decay of (u, v) we obtain a uniform bound on $\|u\|_{L^4(\mathbb{R}^n)}, \|v\|_{L^4(\mathbb{R}^n)}$ which, using the differential equation (1), gives a uniform bound on $\|u\|_{\lambda_1}, \|v\|_{\lambda_2}$. This finishes the proof. $\square$

Let us mention that in view of Proposition 5 the a priori bounds from the above Lemma cannot be extended to the interval $s \in [0, \min\{\frac{\alpha}{\lambda_1}, \frac{\beta}{\lambda_2}\}]$. Furthermore, positive solutions of (1) are not uniformly bounded for parameters s belonging to neighbourhoods of 0 when $n \geq 4$,

see Remark 2. Notice that the assumption $n \in \{1,2,3\}$ in the proof of the above Lemma only becomes important when we apply Theorem 8.4 in [15].

7. Appendix B

In this section we show that in the one-dimensional case the function $F(\cdot,s) : X \to X$ given by (12) is a compact perturbation of the identity for an appriately chosen Banach space X such that $\mathcal{T}_1, \mathcal{T}_2$ are continuous curves in $X \times (0,\infty)$. Let $\sigma \in (0,1)$ be fixed and set $(X, \langle\cdot,\cdot\rangle_X)$ to be the Hilbert space given by

$$X := \big\{(u,v) \in H^1_r(\mathbb{R}) \times H^1_r(\mathbb{R}) : \langle (u,v),(u,v)\rangle_X < \infty\big\},$$
$$\langle (u,v),(\tilde u,\tilde v)\rangle_X := \int_0^\infty e^{2\sigma\mu_1 x}(u'\tilde u' + \mu_1^2 u\tilde u)\,dx + \int_0^\infty e^{2\sigma\mu_2 x}(v'\tilde v' + \mu_2^2 v\tilde v)\,dx$$

where $\mu_1 := \sqrt{\lambda_1}$ and $\mu_2 := \sqrt{\lambda_2}$. One may check that $(X, \langle\cdot,\cdot\rangle_X)$ is a Hilbert space and the subspace $C^\infty_{0,r}(\mathbb{R}) \times C^\infty_{0,r}(\mathbb{R})$ consisting of smooth even functions having compact support is dense in X. We wil use the formula

$$\big((-\Delta+\mu^2)^{-1}f\big)(x) = \frac{\mu}{2}\int_{\mathbb{R}} e^{-\mu|x-y|} f(y)\,dy = \int_0^\infty \mu\Gamma(\mu x,\mu y) f(y)\,dy \tag{18}$$

for all $f \in C^\infty_{0,r}(\mathbb{R})$ and $\mu > 0$ where $\Gamma(x,y) = \frac{1}{2}(e^{-|x-y|} + e^{-|x+y|})$.

Proof of well-definedness: First let us prove the following estimate for all $(u,v) \in X$:

$$\sqrt{\mu_1}|u(r)| \le \|(u,v)\|_X e^{-\sigma\mu_1 r} \quad \text{and} \quad \sqrt{\mu_2}|v(r)| \le \|(u,v)\|_X e^{-\sigma\mu_2 r} \qquad (r \ge 0). \tag{19}$$

It suffices to prove these inequalities for $u, v \in C^\infty_{0,r}(\mathbb{R})$. For such functions we have

$$\mu_1 u(r)^2 \le 2\mu_1 \int_r^\infty |uu'|\,dx \le e^{-2\sigma\mu_1 r}\int_r^\infty e^{2\sigma\mu_1 x}(u'^2 + \mu_1^2 u^2)\,dx \le \|(u,v)\|_X^2 e^{-2\sigma\mu_1 r},$$
$$\mu_2 v(r)^2 \le 2\mu_2 \int_r^\infty |vv'|\,dx \le e^{-2\sigma\mu_2 r}\int_r^\infty e^{2\sigma\mu_2 x}(v'^2 + \mu_2^2 v^2)\,dx \le \|(u,v)\|_X^2 e^{-2\sigma\mu_2 r}.$$

Next, using $u'(0) = v'(0) = 0$ and that u, v have compact support, we obtain

$$\begin{aligned}\int_0^\infty e^{2\sigma\mu_1 x}(u'^2 + \mu_1^2 u^2)\,dx &= \int_0^\infty (e^{2\sigma\mu_1 x}uu')' - 2\sigma\mu_1 e^{2\sigma\mu_1 x}uu' + e^{2\sigma\mu_1 x}u(-u'' + \mu_1^2 u)\,dx \\ &= -2\sigma\mu_1\int_0^\infty e^{2\sigma\mu_1 x}uu'\,dx + \int_0^\infty e^{2\sigma\mu_1 x}u(-u''+\mu_1^2 u)\,dx \\ &\le \sigma\int_0^\infty e^{2\sigma\mu_1 x}(u'^2 + \mu_1^2 u^2)\,dx + \int_0^\infty e^{2\sigma\mu_1 x}u(-u''+\mu_1^2 u)\,dx.\end{aligned}$$

Performing the analogous rearrangements for v then gives for all $u, v \in C^\infty_{0,r}(\mathbb{R})$

$$\|(u,v)\|_X^2 \le \frac{1}{1-\sigma}\int_0^\infty e^{2\sigma\mu_1 x}u(-u''+\mu_1^2 u)\,dx + \frac{1}{1-\sigma}\int_0^\infty e^{2\sigma\mu_2 x}v(-v''+\mu_2^2 v)\,dx. \tag{20}$$

Applying this inequality to $(u,v) = (-\Delta+\mu_1^2)^{-1}(f)\chi_R, (-\Delta+\mu_2^2)^{-1}(g)\chi_R)$ for $f,g\in C^\infty_{0,r}(\mathbb{R})$ and a suitable family $(\chi_R)_{R>0}$ of cut-off functions converging to 1 we obtain

$$\begin{aligned}
&\Big\|\Big((-\Delta+\mu_1^2)^{-1}(f),(-\Delta+\mu_2^2)^{-1}(g)\Big)\Big\|_X^2 \\
&\overset{(20)}{\leq} \frac{1}{1-\sigma}\int_0^\infty e^{2\sigma\mu_1 x}(-\Delta+\mu_1^2)^{-1}(f)(x)f(x)\,dx \\
&+\frac{1}{1-\sigma}\int_0^\infty e^{2\sigma\mu_2 x}(-\Delta+\mu_2^2)^{-1}(g)(x)g(x)\,dx \\
&\overset{(18)}{=} \frac{\mu_1}{1-\sigma}\int_0^\infty\int_0^\infty e^{2\sigma\mu_1 x}\Gamma(\mu_1 x,\mu_1 y)f(x)f(y)\,dx\,dy \\
&+\frac{\mu_2}{1-\sigma}\int_0^\infty\int_0^\infty e^{2\sigma\mu_2 x}\Gamma(\mu_2 x,\mu_2 y)g(x)g(y)\,dx\,dy \\
&\leq \frac{\mu_1}{1-\sigma}\int_0^\infty\int_0^\infty e^{\sigma\mu_1 x}e^{\sigma\mu_1 y}|f(x)||f(y)|\,dx\,dy \\
&+\frac{\mu_2}{1-\sigma}\int_0^\infty\int_0^\infty e^{\sigma\mu_2 x}e^{\sigma\mu_2 y}|g(x)||g(y)|\,dx\,dy \\
&=\frac{\mu_1}{1-\sigma}\Big(\int_0^\infty e^{\sigma\mu_1 x}|f(x)|\,dx\Big)^2+\frac{\mu_2}{1-\sigma}\Big(\int_0^\infty e^{\sigma\mu_2 x}|g(x)|\,dx\Big)^2.
\end{aligned}$$

Plugging in

$$f := f_{u,v} := \frac{\alpha u Z}{1+sZ} \leq \alpha u(\alpha u^2+\beta v^2), \qquad g := g_{u,v} := \frac{\beta v Z}{1+sZ} \leq \beta v(\alpha u^2+\beta v^2)$$

and using the estimate (19) we find that there is a positive number C depending on $\sigma,\mu_1,\mu_2,\alpha,\beta$ but not on u,v such that

$$\Big\|\Big((-\Delta+\mu_1^2)^{-1}(f_{u,v}),(-\Delta+\mu_2^2)^{-1}(g_{u,v})\Big)\Big\|_X \leq C\|(u,v)\|_X^3. \tag{21}$$

By density of $C^\infty_{0,r}(\mathbb{R})\times C^\infty_{0,r}(\mathbb{R})$ in X this inequality also holds for $(u,v)\in X$. If now (u_k,v_k) is a sequence in $C^\infty_{0,r}(\mathbb{R})\times C^\infty_{0,r}(\mathbb{R})$ converging to $(u,v)\in X$ then similar estimates based on (19) show

$$\begin{aligned}
&\Big\|\Big((-\Delta+\mu_1^2)^{-1}(f_{u_k,v_k}-f_{u_m,v_m}),(-\Delta+\mu_2^2)^{-1}(g_{u_k,v_k}-g_{u_m,v_m})\Big)\Big\|_X \\
&\leq C\|(u_k-u_m,v_k-u_m)\|_X(\|(u_k,v_k)\|_X+\|(u_m,v_m)\|_X)^2
\end{aligned}$$

for some $C>0$ implying that $F: X\times(0,\infty)\to X$ is well-defined and that (21) also holds for $(u,v)\in X$.

Proof of compactness of $\mathrm{Id}-F$*:* Let now (u_m,v_m) be a bounded sequence in X. Then we can without loss of generality assume $(u_m,v_m)\rightharpoonup(u,v)\in X$ and $(u_m,v_m)\to(u,v)$ pointwise almost everywhere. We set

$$f_m := \frac{\alpha u_m Z_m}{1+sZ_m}, \qquad g_m := \frac{\beta v_m Z_m}{1+sZ_m}, \qquad f := \frac{\alpha u Z}{1+sZ}, \qquad g := \frac{\beta v Z}{1+sZ}.$$

where $Z_m := \alpha u_m^2 + \beta v_m^2$ and $Z := \alpha u^2 + \beta v^2$. Then we have $f_m \to f, g_m \to g$ pointwise almost everywhere and the estimate (19) implies

$$|f_m(r)| + |f(r)| \le \alpha(|u_m(r)|Z_m(r) + |u(r)|Z(r)) \le C(e^{-3\sigma\mu_1 r} + e^{-\sigma(\mu_1+2\mu_2)r}), \tag{22}$$

$$|g_m(r)| + |g(r)| \le \beta(|v_m(r)|Z_m(r) + |v(r)|Z(r)) \le C(e^{-3\sigma\mu_2 r} + e^{-\sigma(\mu_2+2\mu_1)r}) \tag{23}$$

for some positive number $C > 0$. Using the estimate from above we therefore obtain

$$\begin{aligned}
&\|(\mathrm{Id} - F)(u_m, v_m) - (\mathrm{Id} - F)(u, v)\|_X^2 \\
&= \Big\| \Big((-\Delta + \mu_1^2)^{-1}(f_m - f), (-\Delta + \mu_2^2)^{-1}(g_m - g) \Big) \Big\|_X^2 \\
&\le \frac{\mu_1}{1-\sigma} \Big(\int_0^\infty e^{\sigma\mu_1 x} |f_m(x) - f(x)| \, dx \Big)^2 + \frac{\mu_2}{1-\sigma} \int_0^\infty e^{\sigma\mu_2 x} |g_m(x) - g(x)| \, dx \Big)^2.
\end{aligned}$$

Using (22),(23) and the dominated convergence theorem we finally get

$$\|(\mathrm{Id} - F)(u_m, v_m) - (\mathrm{Id} - F)(u, v)\|_X \to 0 \qquad \text{as } m \to \infty$$

which is all we had to show.

8. Appendix C

Finally we prove a spectral theoretical result which we used in the proof of Proposition 6 and for which we could not find a reference in the literature. The key ingredient of this result is the min-max-principle for eigenvalues of semibounded selfadjoint Schrödinger operators, see for instance Theorem XIII.2 in [17]. As in Proposition 6 we denote by $\mu_k(s)$ $(k \in \mathbb{N}_0)$ the k-th eigenvalue of the compact selfadjoint operator

$$L_s : H_r^1(\mathbb{R}^n) \to H_r^1(\mathbb{R}^n), \quad L_s\phi := (-\Delta + \lambda)^{-1}(W_s\phi) \tag{24}$$

for potentials W_s vanishing at infinity, i.e. $W_s(x) \to 0$ as $|x| \to \infty$.

Lemma 2. *Let $n \in \mathbb{N}$ and $\kappa, \lambda > 0, a < b$, let $(W_s)_{s\in(a,b)}$ be a family of radially symmetric potentials $W_s : \mathbb{R}^n \to [0, \infty)$ vanishing at infinity and satisfying*

$$\text{(i)} \quad \limsup_{s\to b} \|W_s\|_\infty = \kappa \qquad \text{and} \qquad \text{(ii)} \quad W_s \to \kappa \text{ locally uniformly as } s \to b.$$

Then for all $k \in \mathbb{N}_0$ we have $\mu_k(s) \to \frac{\kappa}{\lambda}$ as $s \to b$.

Proof. The min-max-principle and (i) imply

$$\limsup_{s\to b} \mu_k(s) \le \limsup_{s\to b} \frac{\|W_s\|_\infty}{\lambda} = \frac{\kappa}{\lambda}.$$

So it remains to show the corresponding estimate from below. Given the assumptions $W_s \ge 0$ and (ii) we find that it is sufficient to show $\mu_k^\varepsilon \to \frac{\kappa}{\lambda}$ as $\varepsilon \to 0$ where μ_k^ε denotes the k-th eigenvalue of the compact self-adjoint operator $M_\varepsilon : H_r^1(\mathbb{R}^n) \to H_r^1(\mathbb{R}^n)$ defined by $M_\varepsilon\phi = (-\Delta + \lambda)^{-1}((\kappa - \varepsilon)1_{B_{1/\varepsilon}}\phi)$. Here, $1_{B_{1/\varepsilon}}$ denotes the indicator function of the ball in $\mathbb{R}^n$

centered at the origin with radius $1/\varepsilon$. Since $\varepsilon \to M_\varepsilon$ is continuous on $(0,\infty)$ with respect to the operator norm the min-max characterization of the eigenvalues implies that

$$\varepsilon \mapsto \omega_k^\varepsilon \text{ is continuous on } (0,\infty) \quad \text{where} \quad \omega_k^\varepsilon := \frac{\kappa-\varepsilon}{\mu_k^\varepsilon} - \lambda.$$

By definition of $\mu_k^\varepsilon, \omega_k^\varepsilon$ the boundary value problem

$$\begin{aligned} &-\phi''(r) - \frac{n-1}{r}\phi'(r) = \omega_k^\varepsilon \phi(r) && \text{for } 0 \le r \le \varepsilon^{-1}, \\ &-\phi''(r) - \frac{n-1}{r}\phi'(r) + \lambda\phi(r) = 0 && \text{for } r \ge \varepsilon^{-1}, \\ &\phi'(0)=0, \quad \phi(r)\to 0 \ \text{ as } r\to\infty, \quad \phi \in C^1([0,\infty)). \end{aligned}$$

has a nontrivial solution. Testing the differential equation on $[0,\varepsilon^{-1}]$ with ϕ we obtain $\omega_k^\varepsilon > 0$. Hence, ϕ is given by

$$\phi(r) = \alpha \cdot \begin{cases} c \cdot r^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(\sqrt{\omega_k^\varepsilon}\, r) & \text{if } r \le \varepsilon^{-1}, \\ r^{\frac{2-n}{2}} K_{\frac{n-2}{2}}(\sqrt{\lambda} r) & \text{if } r \ge \varepsilon^{-1} \end{cases}$$

for some $\alpha \neq 0$. Here, K denotes the modified Bessel function of the second kind and J represents the Bessel function of the first kind. From $\phi \in C^1([0,\infty))$ we get the following conditions on c and ω_k^ε:

$$K_{\frac{n-2}{2}}(\sqrt{\lambda}\varepsilon^{-1}) = cJ_{\frac{n-2}{2}}(\sqrt{\omega_k^\varepsilon}\,\varepsilon^{-1}), \qquad \sqrt{\lambda}K'_{\frac{n-2}{2}}(\sqrt{\lambda}\varepsilon^{-1}) = \sqrt{\omega_k^\varepsilon} cJ'_{\frac{n-2}{2}}(\sqrt{\omega_k^\varepsilon}\,\varepsilon^{-1}).$$

Due to the continuity of $\varepsilon \to \omega_k^\varepsilon$ on $(0,\infty)$ and due to the fact that K is positive whereas J has infinitely many zeros going off to infinity we infer that $\sqrt{\omega_k^\varepsilon}\varepsilon^{-1}$ is bounded on $(0,\infty)$. In particular this gives $\omega_k^\varepsilon \to 0$ and thus $\mu_k^\varepsilon \to \frac{\kappa}{\lambda}$ as $\varepsilon \to 0$ which is all we had to show. $\square$

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